Three-dimensional coupled-mode theory of free-electron lasers in the collective regime

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An analytical three-dimensional model is presented for free-electron lasers (FELs) operating in the small-signal linear regime. The excitation of radiation and space-charge waves is found by expanding the total electromagnetic field in terms of transverse eigenmodes in a waveguide of arbitrary cross section and solving the evolution of their amplitudes from a set of coupled excitation equations. Coupled-mode theory is employed to derive dispersion relations for the space-charge waves and for the gain. The eigenmodes of the FELs ("supermodes") and the gain for each of them are derived after diagonalization of the coupled-mode system. It is found that for the case of degenerate coupled modes (equal axial wave numbers), the normal modes satisfy the well known FEL gain dispersion equation with a modified gain parameter. The gain of the supermode, calculated according to the presented coupled-mode theory, is higher than the gain of the individual modes if calculated on the basis of a single-mode model. We demonstrate the formalism by finding the gain of the TE_{01}, and the coupled TE_{21} and TM_{21} modes excited simultaneously in a rectangular waveguide.

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I. INTRODUCTION

The well known one-dimensional (1D) linear model of a free-electron laser (FEL) [1–6] assumes a single mode of the signal field and a finite cross section of the electron beam. The coupling between the radiation wave and the beam is proportional to the *power filling factor,* defined as the ratio between the electron beam cross-section area and the effective electromagnetic mode area.

It was shown in [7–9] that a variety of FEL schemes satisfy the same gain-dispersion relation in the small-signal regime. These papers presented a unified analysis valid in the cold and warm electron beams, low and high gain limits, and single-particle (Compton) or collective (Raman) regimes.

In overmoded waveguides and optical open resonators it is necessary to use a more elaborate model of FEL interaction, which includes three-dimensional aspects of the radiation and of the space-charge waves. Several linear and nonlinear analyses were carried out, expanding the transverse radiation field in terms of free-space Hermite-Gaussian modes [10–15]. FEL theories for circular waveguides also exist [16,17]. A linear model developed in Refs. [18–20] is based on representation of the electromagnetic field as a Fourier series of plane waves for which a matrix dispersion relation is found. This method is applicable only for cases where the electromagnetic field is propagating in free space or in rectangular waveguides.

When the FEL is operating in the collective (Raman) regime [21,22], space-charge forces in the electron-beam affect the FEL operation. A 1D description of the space-charge field is inaccurate because of the finite transverse dimensions of the beam and the effect of the conducting waveguide walls. In most of the papers on FELs in the collective regime [19,23–27], the space-charge eignenmodes of the electron-beam [28–31] were found analytically for cases where there was planar, rectangular, or circular symmetry.

A coupled-mode theory of plasma wave excitation in a nonradiating structure was published in [32]. In the present paper we extend the analysis to include excitation and propagation of solenoidal electromagnetic waves as well as space-charge waves in FELs operating in the linear regime. The total electromagnetic (signal and space-charge) field is expressed as a sum of the transverse eigenmodes of the empty, arbitrary cross-section waveguide. Employing the linearized plasma fluid model [33], we derive a matrix form of the dispersion relation, from which the evolution of the space-charge density modulation along the interaction region is found.

A set of coupled gain dispersion equations for the radiation wave is derived. It is shown that in the case that coupled transverse modes are degenerate in their longitudinal wave number k_z, the system can be diagonalized to find the normal modes ("supermodes") of the FEL and the gain of each mode. These eigenmodes of the coupled system are the steady-state transverse modes of the FEL oscillator if the resonator mirrors do not scatter the transverse modes to each other or reflect them selectively. The linear 3D coupled-mode analysis given here can be adopted for a FEL scheme which utilizes a cold electron beam and a wave guide with arbitrary transverse geometries.

II. MODEL EXPANSION OF THE ELETROMAGNETIC FIELD

The total time-harmonic electromagnetic field can be represented in terms of a complete set of eigenfunctions, which are the mode solutions of the empty medium [34–36]. The transverse component of the field is written as a linear superposition of forward ($+q$) and backward ($-q$) transverse modes

$$\tilde{\mathbf{E}}_{\perp}(\mathbf{r})=\sum_{q}[C_{+q}(z)e^{+jk_{zq}z}+C_{-q}(z)e^{-jk_{zq}z}]\tilde{\mathcal{E}}_{q\perp}(x,y)\ ,$$
$$\tilde{\mathbf{H}}_{\perp}(\mathbf{r})=\sum_{q}[C_{+q}(z)e^{+jk_{zq}z}-C_{-q}(z)e^{-jk_{zq}z}]\tilde{\mathcal{H}}_{q\perp}(x,y)\ . \tag{1}$$

$\mathbf{r}$ stands for the (x,y,z) coordinates, where (x,y) are the transverse coordinates and z is the longitudinal axis of propagation. $C_q(z)$ is the slowly varying amplitude and $\tilde{\mathcal{E}}_{q\perp}(x,y)$ and $\tilde{\mathcal{H}}_{q\perp}(x,y)$ are complex vectors representing the transverse electric and magnetic profiles and the polarization of mode q. The summations include propagating and cutoff TE and TM modes, for which k_{zq} is the wave number. (For cutoff modes k_{zq} is an imaginary number with a positive sign coefficient.) The longitudinal components $\tilde{E}_z(\mathbf{r})$ and $\tilde{H}_z(\mathbf{r})$ of the field are derived after substitution of the expansion into the inhomogeneous, steady-state Maxwell equations [35], resulting in

$$\tilde{E}_z(\mathbf{r})=\sum_{q}[C_{+q}(z)e^{+jk_{zq}z}-C_{-q}(z)e^{-jk_{zq}z}]\times\tilde{\mathcal{E}}_{qz}(x,y)+\frac{1}{j\omega\epsilon}\tilde{J}_z(\mathbf{r})\ ,$$
$$\tilde{H}_z(\mathbf{r})=\sum_{q}[C_{+q}(z)e^{+jk_{zq}z}+C_{-q}(z)e^{-jk_{zq}z}]\tilde{\mathcal{H}}_{qz}(x,y)\ . \tag{2}$$

$\tilde{\mathcal{E}}_{qz}(x,y)$ and $\tilde{\mathcal{H}}_{qz}(x,y)$ are the longitudinal electric and magnetic field profiles of the TM mode and of the TE mode, respectively.

Imposing the appropriate boundary conditions, the Maxwell vector field equations are transformed into scalar differential equations, which describe the evolution of the slowly varying amplitude of the forward $(+q)$ mode

$$\frac{d}{dz}C_{+q}(z)=-\frac{1}{2Z_q^*\mathcal{S}_q}e^{-jk_{zq}z}\int\int[Z_q\tilde{\mathbf{J}}_{\perp}(\mathbf{r})+\hat{\mathbf{z}}Z_q^*\tilde{J}_z(\mathbf{r})]\cdot\tilde{\mathcal{E}}_q^*(x,y)dx\,dy \tag{3}$$

and of the backward $(-q)$ mode

$$\frac{d}{dz}C_{-q}(z)=+\frac{1}{2Z_q^*\mathcal{S}_q}e^{+jk_{zq}z}\int\int[Z_q\tilde{\mathbf{J}}_{\perp}(r)-\hat{\mathbf{z}}Z_q^*\tilde{J}_z(\mathbf{r})]\cdot\tilde{\mathcal{E}}_q^*(x,y)dx\,dy\ . \tag{4}$$

Z_q is the mode impedance given by $Z_{\mathrm{TE}q}=\omega\mu/k_{zq}$ for TE modes and $Z_{\mathrm{TM}q}=k_{zq}/\omega\epsilon$ for TM modes. The normalization of the field amplitudes of each mode is done via that mode's complex Poynting vector power

$$\mathcal{S}_q=\int\int[\tilde{\mathcal{E}}_{q\perp}(x,y)\times\tilde{\mathcal{H}}_{q\perp}^*(x,y)]\cdot\hat{z}\,dx\,dy\ . \tag{5}$$

The total power carried in the electromagnetic wave is given by

$$P(z)=\sum_{\substack{q\\ \text{propagating}}}[|C_{+q}(z)|^2-|C_{-q}(z)|^2]\mathcal{P}_q-\sum_{\substack{q\\ \text{cut off}}}\mathrm{Im}\{C_{+q}^*(z)C_{-q}(z)\}\mathrm{Im}\{\mathcal{S}_q\}\ , \tag{6}$$

where $\mathcal{P}_q\equiv\frac{1}{2}\mathrm{Re}\{\mathcal{S}_q\}$ is the normalized power of the propagating mode q.

III. EXCITATION OF THE RADIATION FIELD

In a FEL the longitudinal pondermotive force produces modulation in the axial velocity of electrons. In the small-signal analysis it is assumed that this perturbation oscillates at a single angular frequency $\omega_\iota\equiv\omega_s-\omega_w$ resulting from the "beating" of the signal at frequency ω_s with the wiggler at frequency ω_w. (For a magnetostatic wiggler $\omega_w=0$.) The first-order expansion of the axial velocity of electrons is given by

$$v_z(\mathbf{r},t)=v_{z0}+\mathrm{Re}\{\tilde{v}_{z1}(\mathbf{r})e^{-j\omega_\iota t}\}\ . \tag{7}$$

It is assumed that there are no energy or angular spreads in the beam and the electrons all move with the same average axial velocity v_{z0} (cold beam limit).

Variations in the axial velocity due to the wiggling are presently ignored. Oscillations of half the wiggler's period emerge when the electrons pass through a linear undulator (e.g., magnetostatic planar wiggler) and cause a reduction in the gain of the FEL operating at the fundamental frequency and lasing at higher odd harmonics. However, the results can be modified to consider this effect as pointed out in [9]. The fluid model employed here can be extended to include operation at high harmonics following the analysis presented [37] for single transverse mode excitation.

Along the interaction region, the momentum modulation develops density bunching in the electron beam. The total space-charge density of the electron beam is described in a linear model by

$$n(\mathbf{r},t)=n_0(x,y)+\mathrm{Re}\{\tilde{n}_1(\mathbf{r})e^{-j\omega_\iota t}\}\ , \tag{8}$$

where $n_0(x,y)$ is the dc part of the beam density and $\tilde{n}_1(\mathbf{r})$ is the first-order perturbation of the density modulation oscillating at ω_ι.

Excitation of the signal wave in the FEL is caused by the transverse component of the current which is oscillating at the frequency ω_s in resonance with the signal field. For an undepleted pump, the phasor of the current density is given in terms of the density modulation wave $\tilde{n}_1(\mathbf{r})$ and the wiggling transverse velocity $\tilde{\mathbf{V}}_{\perp}^{w}$ of the electrons

$$\tilde{\mathbf{J}}_{\perp}^{s}(\mathbf{r})=-\tfrac{1}{2}e\tilde{n}_1(\mathbf{r})\tilde{\mathbf{V}}_{\perp}^{w}e^{-jk_wz}\ . \tag{9}$$

Because of the finite extent of the wiggling electron quiver amplitude, the variation of the electric field in the transverse dimension has to be taken into account in the interaction [38]. To find the field distribution at a plane z

along the path of the electrons, we expand the local electric field profile around the axis of the average electron trajectory (x_e, y_e) in a Taylor series:

$$\tilde{\mathcal{E}}_q(x(z),y(z)) \simeq \tilde{\mathcal{E}}_q(x_e,y_e)+(\mathrm{Re}\{\tilde{\mathbf{r}}_{\perp w}e^{-jk_w z}\}\cdot\nabla_\perp)\tilde{\mathcal{E}}_q(x,y)|_{x_e,y_e} , \quad (10)$$

where $\tilde{\mathbf{r}}_{\perp w}=j(\tilde{\mathcal{V}}_\perp^w/k_w v_{z0})$ is the amplitude transverse displacement of the wiggling electron trajectory.

The amplitude of the propagating mode q of the signal field excited in the FEL is found after substitution of those current products, which are phase matched with the mode of profile given by (10), into Eq. (3). [Equation (11) is for the forward mode going in the $+z$ direction. The excitation equation for backward wave interaction is expressed by replacing $\mathcal{P}_q$ by $-\mathcal{P}_q$ and k_{zq} by $-k_{zq}$.]:

$$\frac{d}{dz}C_q(z)=\frac{e}{8\mathcal{P}_q}e^{-j(k_{zq}+k_w)z}\int\int\tilde{n}_1(\mathbf{r})[\tilde{\mathcal{V}}_\perp^w+\hat{z}v_{z0}(\tilde{r}_{\perp w}\cdot\nabla_\perp)]\cdot\tilde{\mathcal{E}}_q^*(x,y)dx\,dy . \quad (11)$$

Since for TM modes $\nabla_\perp\tilde{\mathcal{E}}_{qz}=j(k_{\perp q}^2/k_{zq})\tilde{\mathcal{E}}_{q\perp}$, the excitation equation can be written in the form

$$\frac{d}{dz}C_q(z)=\frac{e\zeta_q}{8\mathcal{P}_q}e^{-j(k_{zq}+k_w)z}\times\int\int\tilde{n}_1(r)\tilde{\mathcal{V}}_\perp^w\cdot\tilde{\mathcal{E}}_{q\perp}^*(x,y)dx\,dy , \quad (12)$$

where

$$\zeta_q=\begin{cases}1 & \text{for TE modes}\\ 1-\dfrac{k_{\perp q}^2}{k_{zq}k_w} & \text{for TM modes .}\end{cases}$$

The evolution of the slowly varying mode amplitude $C_q(z)$ of the signal is observed to be associated with the space-charge density modulation $\tilde{n}_1(\mathbf{r})$. The density modulation is derived from the electron-beam fluid equations in the following.

IV. THE ELECTRON-BEAM FLUID MODEL

The longitudinal ac part of the current density resulting from the density and velocity modulation in the beam is given by

$$\tilde{J}_{z1}(\mathbf{r})=-e[n_0(x,y)\tilde{v}_{z1}(\mathbf{r})+v_{z0}\tilde{n}_1(\mathbf{r})] . \quad (13)$$

The relation between the space-charge density oscillation and the velocity modulation is found from the continuity equation. In confined beams the transverse divergence of the current density is usually sufficiently small compared with the longitudinal one ($|\nabla_\perp\cdot\tilde{\mathbf{J}}_{\perp 1}|\ll|\partial\tilde{J}_{z1}/\partial z|$). Under this approximation the equation of continuity can be written

$$\frac{d\tilde{J}_{z1}(\mathbf{r})}{dz}=-j\omega_\iota e\tilde{n}_1(\mathbf{r}) . \quad (14)$$

The axial velocity $\tilde{v}_{z1}(\mathbf{r})$ is found from the relativistic axial force equation written in its small-signal form

$$\frac{d}{dz}\tilde{v}_{z1}(\mathbf{r})-j\frac{\omega}{v_{z0}}\tilde{v}_{z1}(r)=-\frac{e}{\gamma_0\gamma_{z0}^2 m v_{z0}}[\tilde{E}_{\mathrm{pond}}(\mathbf{r})+\tilde{E}_z^{\mathrm{SC}}(\mathbf{r})] . \quad (15)$$

$\gamma_0=(1-\beta_0^2)^{-1/2}$ is the Lorentz factor and $\gamma_{z0}\equiv(1-\beta_{z0}^2)^{-1/2}$. The forcing term on the right-hand side of Eq. (15) consists of the pondermotive field $\tilde{E}_{\mathrm{pond}}(\mathbf{r})$ and the longitudinal component of the space-charge field $\tilde{E}_z^{\mathrm{SC}}(\mathbf{r})$.

Each waveguide mode interacts independently with the transverse components of electron velocity to produce a longitudinal pondermotive field

$$\tilde{E}_{\mathrm{pond}}(\mathbf{r})=\sum_q C_q(z)\tilde{\mathcal{E}}_q^{\mathrm{pm}}(x,y)e^{+j(k_{zq}+k_w)z} , \quad (16)$$

where we define $\tilde{\mathcal{E}}_q^{\mathrm{pm}}(x,y)\equiv\frac{1}{2}[\tilde{\mathcal{V}}_{\perp q}\times\tilde{\mathcal{B}}_\perp^{w*}+\tilde{\mathcal{V}}_\perp^{w*}\times\tilde{\mathcal{B}}_{\perp q}]\cdot\hat{\mathbf{z}}$ [8]. The longitudinal space-charge field $\tilde{E}_z^{\mathrm{SC}}(\mathbf{r})$ is a result of the density bunching in the beam and is excited by the longitudinal current density $\tilde{J}_{z1}(\mathbf{r})$ given in Eq. (13).

The moment equations (13)–(16) can be combined with the expression for the longitudinal electric field $\tilde{E}_z^{\mathrm{SC}}(\mathbf{r})=\tilde{E}_z(\mathbf{r})$ given in Eq. (2), resulting in a differential equation of second order for the density bunching:

$$\frac{d^2}{dz^2}\tilde{n}_1(\mathbf{r})-2j\frac{\omega}{v_{z0}}\frac{d}{dz}\tilde{n}_1(\mathbf{r})+\frac{\omega_p^2(x,y)-\omega_\iota^2}{v_{z0}^2}\tilde{n}_1(\mathbf{r}) = j\frac{\omega_p^2(x,y)}{v_{z0}^2}\frac{\epsilon_0}{e}\left[\sum_q(k_{zq}+k_w)C_q(z)\tilde{\mathcal{E}}_q^{\mathrm{pm}}(x,y)\times e^{+j(k_{zq}+k_w)z}+\sum_i k_{zi}V_i(z)\tilde{\mathcal{E}}_{iz}(x,y)\right] . \quad (17)$$

$\omega_p^2(x,y)\equiv(e^2/\gamma_0\gamma_{z0}^2\epsilon_0 m)n_0(x,y)$ and $V_i(z)$ is the fast varying amplitude of mode i excited by the longitudinal current density $\tilde{J}_{z1}(\mathbf{r})$, given by

$$V_i(z)\equiv C_{+i}(z)e^{+jk_{zi}z}+C_{-i}(z)e^{-jk_{zi}z}$$

and satisfies the excitation equation

$$\frac{d^2}{dz^2}V_i(z)+k_{zi}^2V_i(z)=j\omega_\iota e\frac{1}{\mathcal{S}_i}\int\int \tilde{n}_1(\mathbf{r})\tilde{\mathcal{E}}_{iz}^*(x,y)dxdy \; . \tag{18}$$

The evolution of the density modulation in the electron beam along the interaction region is given by Eq. (17) and the set of equations (18). The FEL interaction is fully described by taking into account also the excitation of the signal field, given in Eq. (12).

V. DISPERSION OF THE SPACE-CHARGE WAVES EXCITED IN THE FEL

The dispersion equation for the density modulation in the beam is found by a Laplace transformation in the z variable of the FEL equations. Assuming that there are no prebunching effects, i.e., $\tilde{n}_1(z=0)=\tilde{v}_{z1}(z=0)=\tilde{E}_{\rm SC}(z=0)=0$, the amplitudes $V_i(s)$ from Eq. (18) are substituted into Eq. (17):

$$\left[\left[s-j\frac{\omega_\iota}{v_{z0}}\right]^2+\frac{\omega_p^2(x,y)}{v_{z0}^2}\right]\tilde{n}_1(s,x,y)+\frac{\omega_p^2(x,y)}{v_{z0}^2}\sum_i\frac{k_{\perp i}^2}{s^2+k_{zi}^2}\tilde{\mathcal{E}}_{iz}(x,y)\frac{\int\int\tilde{n}_1(s,x,y)\tilde{\mathcal{E}}_{iz}^*(x,y)dxdy}{\int\int|\tilde{\mathcal{E}}_{iz}(x,y)|^2dxdy}$$
$$=j\frac{\epsilon_0}{e}\frac{\omega_p^2(x,y)}{v_{z0}^2}\sum_q(k_{zq}+k_w)\mathcal{C}_q[s-j(k_{zq}+k_w)]\tilde{\mathcal{E}}_q^{\rm pm}(x,y) \; . \tag{19}$$

The electric field profile functions $\tilde{\mathcal{E}}_{iz}(x,y)$ of the longitudinal component of the TM modes including cutoff modes are a complete set of orthogonal functions that can be used to expand the density bunching in a linear combination

$$\tilde{n}_1(s,x,y)=\sum_i A_i(s)\tilde{\mathcal{E}}_{iz}(x,y) \; . \tag{20}$$

Introducing this expansion into Eq. (19) and multiplying both sides by $\tilde{\mathcal{E}}_{i'z}^*(x,y)$ gives

$$\left[s-j\frac{\omega_i}{v_{z0}}\right]^2A_{i'}(s)+\sum_i A_i(s)\theta_{pi'i}^2(s)$$
$$=j\frac{\epsilon_0}{e}\sum_q(k_{zq}+k_w)\mathcal{C}_q[s-j(k_{zq}+k_w)]$$
$$\times\int\int\frac{\omega_p^2(x,y)}{v_{z0}^2}\tilde{\mathcal{E}}_q^{\rm pm}(x,y)$$
$$\times\tilde{\mathcal{E}}_{i'z}^*(x,y)dx\,dy \; , \tag{21}$$

where

$$\theta_{pi'i}^2(s)\equiv\left[1+\frac{k_{\perp i}^2}{s^2+k_{zi}^2}\right]\times\frac{\int\int\frac{\omega_p^2(x,y)}{v_{z0}^2}\tilde{\mathcal{E}}_{iz}(x,y)\tilde{\mathcal{E}}_{i'z}^*(x,y)dx\,dy}{\int\int|\tilde{\mathcal{E}}_{i'z}(x,y)|^2dx\,dy} \; .$$

By taking the expression of the slowly varying amplitudes $\mathcal{C}_q(s)$ for the electromagnetic modes of the signal wave from Eq. (12) and substituting into Eq. (21) a set of algebraic equations for the expansion coefficients $A_i(s)$ of the density modulation is derived:

$$s\left[\left[s-j\frac{\omega_i}{v_{z0}}\right]^2A_{i'}(s)+\sum_i A_i(s)\theta_{pi'i}^2(s)\right]$$
$$-j\sum_q\sum_i A_i(s)\mathcal{Q}_{i'iq}=\sum_q C_q(z=0) \; . \tag{22}$$

The signal field whose modal expansion amplitudes at the entrance to the interaction region are $C_q(0)$ generates modulation in the space-charge density of the beam. The parameter

$$\mathcal{Q}_{i'iq}\equiv\frac{\epsilon_0\zeta_q}{8\mathcal{P}_q}(k_{zq}+k_w)$$
$$\times\int\int\frac{\omega_p^2(x,y)}{v_{z0}^2}\tilde{\mathcal{E}}_q^{\rm pm}(x,y)\tilde{\mathcal{E}}_{i'z}(x,y)dx\,dy$$
$$\times\int\int\tilde{\mathcal{E}}_{iz}(x,y)\nabla_\perp^w\cdot\tilde{\mathcal{E}}_{\perp q}^*(x,y)dx\,dy$$

expresses the coupling between the plasma modes in the beam and the electromagnetic modes of the signal.

The inhomogeneous set of equations (22) for the coefficients $A_i(s)$ can be presented in a compact matrix form

$$\left\{s\left[\left[s-j\frac{\omega_\iota}{v_{z0}}\right]^2\underline{\underline{I}}+\underline{\underline{\Theta}}_p^{(2)}(s)\right]-j\sum_q\underline{\underline{\mathcal{Q}}}_q\right\}\underline{A}(s)=\sum_q C_q(0)\underline{I} \; . \tag{23}$$

With no signal wave injected into the system [i.e., $C_q(0)=0$], the above system reduces to a homogeneous set from which the propagation constants $\beta=\mathrm{Im}\{s\}$ of the plasma waves can be found [32]. Since these waves propagate with a velocity that is nearly equal to that of the beam, the matrix $\underline{\underline{\Theta}}_p^{(2)}(s)$ can be approximated by substituting $s\approx j(\omega_\iota/v_{z0})$. The matrix elements are given then by the constants

$$\theta_{pi'i}^2\equiv\frac{1}{1+\left[\gamma_{z0}\beta_{z0}\frac{k_{\perp i}}{k_\iota}\right]^2}\times\frac{\int\int\frac{\omega_p^2(x,y)}{v_{z0}^2}\tilde{\mathcal{E}}_{iz}(x,y)\tilde{\mathcal{E}}_{i'z}^*(x,y)dx\,dy}{\int\int|\tilde{\mathcal{E}}_{i'z}(x,y)|^2dx\,dy} \; ,$$

where $k_\iota=\omega_\iota/c$.

VI. COUPLED-MODE GAIN-DISPERSION EQUATION

The space-charge density modulation (20) in the beam excites the electromagnetic modes of the signal according to (12). Calculation of the gain requires one to first solve for the expansion coefficients $A_i(s)$ of the density modulation. An alternative way is to assume an interaction with a single plasma mode (usually the fundamental one), calculate the plasma frequency reduction factor of this mode, and then modify the FEL interaction equations to take into account the 3D collective effects. Following this latter approach (which is common in electron-tube analysis [26,30]), we derive a set of coupled differential equations for the slowly varying amplitude of the transverse modes excited in the FEL:

$$\frac{d^3}{dz^3}C_q(z)-2j\theta_q\frac{d^2}{dz^2}C_q(z)+(\theta_{pr}^2-\theta_q^2)\frac{d}{dz}C_q(z) = j\sum_{q'}Q_{qq'}C_{q'}(z)e^{-j\Delta k_{zqq'}z} . \quad (24)$$

Here we use the following parameter rotation: (i) the detuning parameter $\theta_q \equiv \omega_s/v_{z0}-(k_{zq}+k_w)$; (ii) the reduced space-charge parameter $\theta_{pr}\equiv\bar{r}\theta_p$, where $\theta_p^2\equiv\omega_p^2/v_{z0}^2=n_0e^2/\gamma_z^2\gamma m\epsilon_0 v_{z0}^2$ is the space-charge parameter of a uniformly distributed electron-beam used in a 1D model and $\bar{r}$ is the plasma frequency reduction factor (which can be calculated using the coupled-mode approach described in [32]); (iii) the gain parameter $Q_{qq'}\equiv\kappa_{qq'}\theta_p^2$, where the coupling parameter is given by

$$\kappa_{qq'}=\frac{\epsilon_0\zeta_q}{8\mathcal{P}_q}(k_{zq}+k_w) \times\int\int f(x,y)\tilde{\mathcal{E}}_{q'}^{pm}(x,y)\tilde{\mathcal{V}}_{\perp}^{w}\cdot\tilde{\mathcal{E}}_{\perp q}^{*}(x,y)dxdy , \quad (25)$$

where $f(x,y)$ is the electron-beam profile. (In the case of a strong planar wiggler, the gain parameter should be modified by multiplying $Q_{qq'}$ by the factor $[JJ]=[J_0(\alpha)-J_1(\alpha)]^2$, where J_0,J_1 are the zero- and first-order Bessel functions, respectively, and $\alpha\equiv(a_w^2/8\gamma^2\beta_{z0}k_w)\omega_s/v_{z0}$, where $a_w=eB_w/mck_w$ [9]) and (iv) $\Delta k_{zqq'}\equiv k_{zq}-k_{zq'}$.

The gain dispersion equation results from a Laplace transformation in the z variable of the FEL interaction equations. Assuming that there are no initial conditions associated with prebunching of the electron beam, one can derive the gain-dispersion equation for the qth propagating waveguide mode of the signal:

$$\mathcal{C}_q(s)=G_{qq}(s)C_q(z=0)+\sum_{q'\neq q}G_{qq'}(s)\mathcal{C}_{q'}(s+j\Delta k_{zqq'}) . \quad (26)$$

$G_{qq}(s)$ is the well known single-mode gain dispersion relation developed previously in [7,8]:

$$G_{qq}(s)\equiv\frac{(s-j\theta_q)^2+\theta_{pr}^2}{s[(s-j\theta_q)^2+\theta_{pr}^2]-jQ_{qq}} . \quad (27)$$

This term describes the amplitude growth of mode q due to self-excitation. The mutual interaction between this mode and other ($q'\neq q$) waveguide modes is expressed by

$$G_{qq'}(s)\equiv\frac{jQ_{qq'}}{s[(s-j\theta_q)^2+\theta_{pr}^2]-jQ_{qq}} . \quad (28)$$

The modes that need to be taken into account in the gain calculation are those that are nearly phase matched to each other and interact efficiently with the electron beam. These modes can be identified by inspection of their single-mode gain curves and observation of overlap at some frequencies. In the case of a finite set of modes that exhibit gain or less at the same frequency, the coupled-mode dispersion equations (26) can be presented in a compact matrix form

$$\underline{\mathcal{C}}(s)=\underline{\underline{\Gamma}}(s)\underline{C}(z=0) . \quad (29)$$

The growth of waveguide mode amplitudes can now be expressed in terms of their initial values $\underline{C}(z=0)$ at the entrance to the interaction region and the gain dispersion matrix $\underline{\underline{\Gamma}}(s)$.

It should be noted that the waveguide modes are not in general the normal modes of the FEL system. Namely, if one starts with a certain transverse mode at $z=0$, it couples to other modes and does not keep its transverse electromagnetic field profile along the interaction length. One can seek a new set of independent modes that are eigenmodes of the FEL system and, except for magnitudes and phases, their field profile does not depend on the propagation coordinate z.

VII. THE FEL SUPERMODES

We now consider the case of coupled waveguide modes which are degenerate in their longitudinal wave number k_{zq}. This is especially applicable in a FEL using a rectangular waveguide where a number of excited waveguide modes may be phase matched to each other and have the same detuning parameter θ. The coupled differential equations (24) can then be written in a simple matrix form

$$\frac{d^3}{dz^3}\underline{C}(z)-2j\theta\frac{d^2}{dz^2}\underline{C}(z)+(\theta_{pr}^2-\theta^2)\frac{d}{dz}\underline{C}(z)=\underline{\underline{Q}}\underline{C}(z) . \quad (30)$$

Mode coupling is expressed by matrix $\underline{\underline{Q}}$, which consists of gain parameters $Q_{qq'}$ defined previously.

The derivation of FEL normal modes is a problem of finding the eigensolutions of coupled differential equations [39,40]. In every plane along the waveguide, any normal mode of the FEL system can be written as a superposition of the waveguide modes. The two representations can be related at every plane through a linear matrix transformation

$$\underline{C}(z)=\underline{\underline{T}}\underline{U}(z) . \quad (31)$$

This transformation, together with Eq. (30), is used to derive a new set of differential equations for the slowly varying amplitudes $U_i(z)$:

$$\frac{d^3}{dz^3}\underline{U}(z)-2j\theta\frac{d^2}{dz^2}\underline{U}(z)+(\theta_{pr}^2-\theta^2)\frac{d}{dz}\underline{U}(z) = \underline{\underline{T}}^{-1}\underline{\underline{Q}}\,\underline{\underline{T}}\,\underline{U}(z) . \quad (32)$$

If the similarity transformation $\underline{\underline{T}}^{-1}\underline{\underline{Q}}\,\underline{\underline{T}}$ produces a diagonal matrix on the right-hand side of Eq. (32), then it becomes a complete set of uncoupled equations and $U_i(z)$ is the slowly varying amplitude of the ith FEL supermode. The diagonal matrix elements are the eigenvalues λ_i of the gain parameter matrix $\underline{\underline{Q}}$ and fulfill the algebraic equation $|\underline{\underline{Q}}-\lambda\underline{\underline{I}}|=0$. The column vectors $\underline{t}_i$ in the matrix $\underline{\underline{T}}$ are the eigenvectors resulting form $\underline{\underline{Q}}\underline{t}_i=\lambda_i\underline{t}_i$.

The dispersion relation for the slowly varying amplitudes of the normal mode i is found after a Laplace transformation of Eq. (32). If there is no prebunching of the electron beam, the initial conditions at the entrance of the FEL interaction region are $d^2U_i(z=0)/dz^2 = dU_i(z=0)/dz=0$, the gain dispersion relation is found directly:

$$\Lambda_i(s)\equiv\frac{\mathcal{U}_i(s)}{U_i(z=0)}=\frac{(s-j\theta)^2+\theta_{pr}^2}{s[(s-j\theta)^2+\theta_{pr}^2]-j\lambda_i} . \quad (33)$$

The dispersion relation (33) for the FEL normal modes resembles the single-mode gain dispersion equation (27), except for the gain parameters λ_i, which are the eigenvalues of matrix $\underline{\underline{Q}}$.

VIII. TWO-MODE COUPLING

We shall demonstrate the coupled-mode formalism described in the preceding section with an example of a waveguide FEL in which only two degenerate modes are excited. In this case the set (30) consists of two equations which are coupled through a 2×2 gain parameter matrix $\underline{\underline{Q}}$, and two supermodes need to be identified. The eigenvalues of the matrix $\underline{\underline{Q}}$ are found first from a quadratic determinantal equation

$$\lambda_{1,2}=\tfrac{1}{2}[Q_{11}+Q_{22}\pm\sqrt{(Q_{11}-Q_{22})^2+4Q_{12}Q_{21}}] . \quad (34)$$

These are the FEL normal mode gain parameters needed in Eq. (33) to express their gain dispersion relation.

The relation between the slowly varying amplitudes of the waveguide modes and the FEL normal modes is expressed by the transformation $\underline{\underline{T}}$, which contains the eigenvectors of $\underline{\underline{Q}}$ in its columns:

$$\begin{bmatrix} C_1(z) \\ C_2(z) \end{bmatrix} = \begin{bmatrix} Q_{12} & \lambda_2-Q_{22} \\ \lambda_1-Q_{11} & Q_{21} \end{bmatrix} \begin{bmatrix} U_1(z) \\ U_2(z) \end{bmatrix} . \quad (35)$$

Note that one of the eigenvector elements is determined arbitrarily.

For the special case of coupled waveguide modes that have nearly the same transverse profiles and polarizations at the position of the electron beam (for example, when the electron-beam cross section is much smaller than the transverse dimensions of the modes), it can be easily shown from (25) that $Q_{11}Q_{22}=Q_{12}Q_{21}$. The resulting eigenvalues in this case are found to be $\lambda_1=Q_{11}+Q_{22}$ and $\lambda_2=0$. One of the supermodes has a gain that is higher than each waveguide mode gain if calculated from a single-mode gain analysis.

IX. NUMERICAL RESULTS

Figure 1 displays an example of coupled-mode gain curve calculations corresponding to the parameters of the Israeli electrostatic accelerator free-electron maser (FEM) [41], designed to lase at millimeter and submillimeter wavelengths. The FEM is based on a 2–6 MeV Tandem Van de Graaff accelerator for a 1-A electron beam and utilizes a magnetostatic planar wiggler with $N_w=20$ periods of $\lambda_w=4.4$ cm. The rf cavity considered in the calculations is an overmoded 1.5×1.5 cm^2 rectangular waveguide.

The modes found to be within the frequency range of operation of the FEM providing gain are the TE_{01}, TE_{21}, and TM_{21} modes. Since the TE_{21} and TM_{21} modes are degenerate in their longitudinal wave number k_{z21}, both can be excited simultaneously and they will have the same frequency. These modes are coupled by the electron-beam of finite cross section and it is thus necessary to use coupled-mode theory to find the mutual gain accurately. Other, nonsynchronous, modes will not participate in the interaction with the electron-beam and need not be taken into account in the coupled-mode equations.

Figure 1 illustrates the gain curves of the TE_{01}, TE_{21}, and TM_{21} waveguide modes as a function of the operating frequency. A distinction is made between the results obtained from single-mode gain calculations (dashed lines) and the more accurate multimode analysis (solid line). Coupled-mode theory was used to calculate the gain parameter of the supermode of the degenerate TE_{21} and TM_{21} modes. The supermode of the planar wiggler FEL is basically the linearly polarized mode LP_{21} of the rectangular waveguide, whose small-signal gain is observed to be higher than each of the separate modes calculated from a single-mode model.

We have also performed gain calculations of the (Electron Laser Facility) (ELF) experiment carried out at the

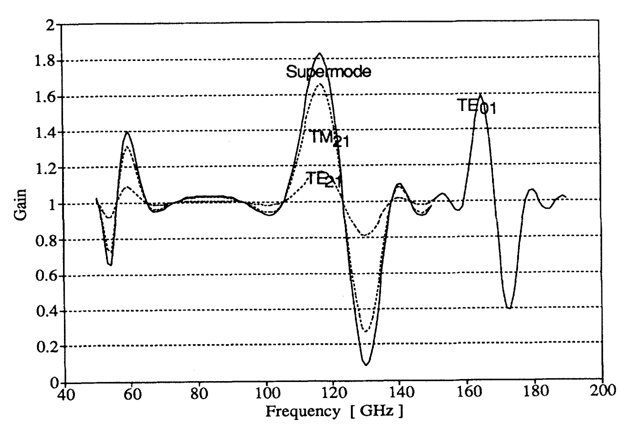

FIG. 1. Small-signal gain curves of the Israeli Tandem free-electron maser.

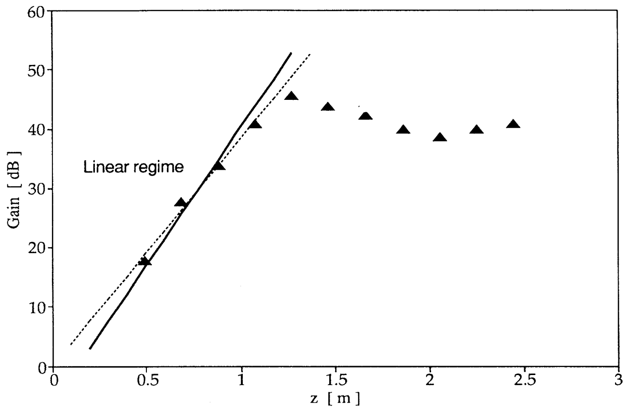

FIG. 2. Comparison of the coupled-mode growth rate calculation with data from the ELF experiments at LLNL.

Lawrence Livermore National Laboratory–Lawrence Berkeley Laboratory, University of California (LLNL-LBL) [42,43]. The ELF utilizes a 3–3.6-MeV, 850-A pulsed electron beam, passing through a linearly polarized electromagnetic wiggler with a period of $\lambda_w = 9.8$ cm. The rf cavity was an oversized 10×3 cm^2 rectangular waveguide. Since the parameters of the ELF experiment are such that the resonant frequency of the degenerate TE_{21} and TM_{21} modes is sufficiently close to the resonant frequency of the fundamental TE_{01} mode, strong coupling between these modes to the electron beam was expected at 34.6 GHz. Experiments showed that the TE_{21} and TM_{21} modes have power levels comparable to the fundamental mode. The ELF experiment operated in the high gain space-charge dominated regime where exponential growth rate was measured.

In Fig. 2 we make a comparison of the theoretical small-signal gain calculations carried out employing the coupled-mode theory (solid line) with the ELF experimental results (filled triangles). The results obtained from the theory and those measured when the ELF amplifier operated in the linear regime are found to be in good agreement. The calculated growth rate is close to the rate measured in the experiment (a linear fit of the experimental results in the small-signal regime appears as a dashed line).

X. CONCLUSIONS

This paper presents a coupled-mode approach for analysis of FEL operation, which takes into account 3D effects of the radiation field and space-charge field. The total electromagnetic field is expanded in terms of the eigenmodes of the waveguide. We derived matrix dispersion relations for the space-charge field and for the signal wave excited in the FEL. From the solution of the coupled gain dispersion equations we attain the small-signal gain curves.

Diagonalization of the coupled modes system yields the eigenmode solutions (supermodes) of the FEL. It is found that the supermode gain obtained for degenerate modes is higher than the gain of the separate modes when calculated in a model where coupling is neglected.

The presented model provides a description of FEL interaction for any kind of symmetry of electron beam and waveguide cross section and can be used to calculate the small-signal gain of FELs operating in the Compton or Raman regimes.

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